

Convergence to Equilibrium for the Discrete Coagulation-Fragmentation Equations with Detailed Balance

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Received: 27 February 2007 / Accepted: 24 June 2007 / Published online: 7 August 2007
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Abstract Under the condition of detailed balance and some additional restrictions on the size of the coefficients, we identify the equilibrium distribution to which solutions of the discrete coagulation-fragmentation system of equations converge for large times, thus showing that there is a critical mass which marks a change in the behavior of the solutions. This was previously known only for particular cases as the generalized Becker–Döring equations. Our proof is based on an inequality between the entropy and the entropy production which also gives some information on the rate of convergence to equilibrium for solutions under the critical mass.

1 Introduction

The discrete coagulation-fragmentation equations (or *DCF equations* for short) are a well-known model for physical processes where a large number of units can join to form groups of two or more. These equations and their continuous version have been studied extensively in recent years in the mathematical and physical literature; as the amount of works dedicated to them is large, we refer to the classical review [14] and the more recent [2, 18] for an overall picture of the field, while more detailed references related to the object of this paper are given below.

The discrete coagulation-fragmentation equations are:

$$\frac{d}{dt}c_j = \frac{1}{2} \sum_{k=1}^{j-1} W_{j-k,k} - \sum_{k=1}^{\infty} W_{j,k}, \quad j \geq 1, \quad (1)$$

where

$$W_{i,j} := a_{i,j}c_i c_j - b_{i,j}c_{i+j}, \quad i, j \geq 1. \quad (2)$$

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Here the unknowns are the functions $c_i = c_i(t)$ for $i \geq 1$, which depend the time $t \geq 0$, and $a_{i,j}, b_{i,j}$ are nonnegative numbers, the *coagulation* and *fragmentation coefficients*, respectively. In the following they are always assumed to be nonnegative and symmetric in i, j . The sum $\sum_{i \geq 1} i c_i(t)$ is usually called the *mass of the solution at time t* , as suggested by the usual physical interpretation of these equations.

One of the long-standing questions has to do with the long time behavior of this system, which is expected to model certain phase change transitions or crystallization processes [7, 15, 19–21]: under some usual conditions it has been proved that there is a *critical mass* ρ_s which marks a qualitative difference in the behavior of the solutions:

- If a solution $\{c_i\}$ has mass above ρ_s , then the solution *converges weakly* to the only equilibrium distribution with mass equal to ρ_s (meaning that each individual $c_i(t)$ converges to the corresponding value for large t). In this case there is loss of mass in infinite time, as the mass of the solution is strictly higher than the mass of the limit distribution.
- If the solution has mass equal to or below ρ_s , then it *converges strongly* to a unique equilibrium distribution, determined by its mass, in the sense that in addition to each c_i , its average cluster size also converges to its equilibrium value. Here mass is also conserved in the limit, as the mass of the solution is the same as the mass of the limit distribution.

Mathematical proofs of this were first given for the Becker–Döring system of equations [4, 6] (which is the particular case of the DCF equations (1) obtained by setting $a_{i,j} = b_{i,j} = 0$ whenever both i and j are greater than 1) and then extended to the generalized Becker–Döring equations [8, 11, 13] (obtained by setting $a_{i,j} = b_{i,j} = 0$ whenever both i and j are greater than some fixed N). For the continuous coagulation-fragmentation equations, a proof of weak convergence to an equilibrium under analogous conditions was given in [17], but to our knowledge there are no available results on the identification of the concrete equilibrium to which a solution converges in the continuous setting.

In this paper we show that the same kind of behavior takes place for the full DCF equations (1) under some conditions on the coefficients $a_{i,j}, b_{i,j}$ which allow, for example, the following case, which is physically representative [5, 11]:

$$a_{i,j} := C(i^\lambda + j^\lambda), \quad (3)$$

$$b_{i,j} := C(i^\lambda + j^\lambda) \exp(C'((i+j)^\mu - i^\mu - j^\mu)), \quad (4)$$

with $0 \leq \lambda \leq 1$, $0 < \mu < 1$ and some constants $C, C' > 0$. More explicitly, we show the following:

Theorem 1.1 *Let c be a solution of the DCF equations (1) under hypotheses 1–6 below; call ρ its mass and ρ_s the critical mass.*

- *If $\rho > \rho_s$ then c converges weakly to the only equilibrium with mass ρ_s .*
- *If $\rho \leq \rho_s$, then c converges strongly to the only equilibrium with mass ρ :*

See Sect. 4 for a more detailed description of the allowed coefficients, a more complete statement of the theorem, and its proof. The hypotheses under which we show this result are specified below as hypotheses 1–6, and Sect. 2 includes a description of the equilibria. We also prove in Sect. 6 an explicit rate of convergence to equilibrium, which is most likely not optimal and is mainly given as a direct consequence of one of the inequalities used in the proof:

Theorem 1.2 *In the above conditions, if the mass ρ of the solution is strictly less than ρ_s , then for some constant C depending only on the coefficients $a_{i,j}, b_{i,j}$, the mass ρ , and the moment of order $2 - \lambda$ of c at time $t = 0$, it holds that*

$$\sum_{i \geq 1} i |c_i - c_i^{eq}| \leq \frac{C}{\sqrt{1 + \log(1 + t)}} \quad \text{for all } t > 0, \tag{5}$$

where $\{c_i^{eq}\}_{i \geq 1}$ is the equilibrium distribution with mass ρ .

The main interest of our result is that it identifies the limiting equilibrium for a general class of coefficients for which all of $a_{i,j}, b_{i,j}$ are nonzero; as explained above, previous results in [4, 6, 8, 11, 13] always imposed that $a_{i,j}, b_{i,j}$ should be zero whenever both i and j are greater than a fixed N . Our statement extends the corresponding ones in [4, 8, 11] except for the fact that we impose a more restrictive condition on the initial data, namely, that it has a finite moment of a certain order which is less than two in common examples. In turn, as explained above, we allow for coefficients in which none of $a_{i,j}, b_{i,j}$ are zero. There are also some restrictions on the coefficients: for example, our result does not apply when, for some $C > 0$,

$$a_{i,j} := C(i^\alpha j^\beta + i^\beta j^\alpha) \tag{6}$$

with α, β both strictly positive (note that the case studied in this paper corresponds to $\beta = \lambda, \alpha = 0$). As mentioned above, our main result is based on Hypotheses 1–6 below; Hypothesis 5, which roughly states that the strength of small-large interactions is comparable to that of large-large interactions, does not hold for this $a_{i,j}$, and whether the result is true also in this case is an open question.

The method of proof of this behavior contained in previous works is based, as a first step, on the study on an *entropy functional* for this equation: a quantity which is decreasing along solutions and allows one to conclude that every solution converges weakly to a suitable equilibrium; and as a second step, on the development of estimates on the amount of large particles by means of which one can identify the average cluster size of the equilibrium to which the solution converges.

Our proof employs the entropy functional for the second step in a new way, based on an inequality between it and its derivative, the *entropy production*, and also a simpler related inequality. As far as we know, this technique has not been previously employed for identifying the mass of the equilibrium to which solutions converge. This approach is inspired in a paper by Jabin and Niethammer [16], where they prove a similar inequality to study the rate of convergence to equilibrium of the Becker–Döring equations. The argument draws strongly from the entropy-entropy production method which was successfully employed to study the long-time behavior of the Aizenman–Bak model [1] (i.e., the continuous coagulation-fragmentation equations with constant coefficients), its inhomogeneous version [12] and other kinetic equations, notably the Boltzmann equation, for which the literature is quite abundant: we refer the reader to the review [3] for further references and background information.

The rough idea is as follows: we use a *relative energy* functional which we denote by F_z and which, as is well known, is decreasing along solutions of our equation (this functional plays the role of entropy here, but is more appropriately named relative energy in agreement with previous uses and its common physical interpretation). Its derivative, which is negative, is called the *free energy dissipation*, denoted by D_{CF} . If the relative energy functional is chosen appropriately, it measures how far a solution is from equilibrium; hence, if one can

show an inequality relating F_z and D_{CF} , one can deduce from that a differential inequality for the evolution of F_z , and with it an estimate on the approach of the solution to equilibrium. As mentioned before, this has been carried out in [16] to estimate the rate of convergence to equilibrium of the Becker–Döring equations. Concretely, the inequality proved there states that for a solution with mass below the critical one and when c_1 is less than a certain critical value,

$$F_z \leq CD \left| \log \frac{1}{F_z} \right|^2, \quad (7)$$

where D is the dissipation rate (which is just D_{CF} is the particular case of the Becker–Döring equations) and C is a constant which depends on the mass of the solution, the distance of c_1 to the critical value mentioned above, and also on a uniform estimate on certain exponential moments of the solution. If one wants to prove Theorem 1.1, the latter estimate is out of reach, as proving such an estimate would automatically yield the result. Hence, our intention is to find a weaker inequality which does not require such a strong estimate of moments of the solution but still allows us to recover useful information on its long-time behavior. It turns out that one can find such an inequality: in Sect. 5 we show that, under conditions similar to the ones above but *without assuming boundedness of exponential moments*, and for some constant C ,

$$F \leq C \sqrt{D_{CF}} \sqrt{M_{2-\lambda}} \quad (8)$$

where F is a slight modification of the relative energy functional (see Sect. 5) and $M_{2-\lambda}$ is the moment of order $2 - \lambda$ of the solution. An easy estimate on this moment shows that it increases at most linearly, and this is enough to deduce Theorem 1.1. Though one could actually deduce it this way, instead we use a simpler inequality which is suggested by this idea and directly gives an estimate on the difference to the equilibrium mass (see Sect. 3).

Aside from being the key point in the proof of our results on the asymptotic behavior of the DCF equations, the inequality we prove below is interesting in itself, as it can be used to deduce other properties of the coagulation–fragmentation system of equations, or to obtain stronger inequalities in other cases. As an application, we give an estimate on the rate of convergence to equilibrium of solutions with mass below the critical one, which is however by no means expected to be optimal; in fact, one would expect the solution to converge to the equilibrium at a rate similar to the one obtained by Jabin and Niethammer in [16], this is, a convergence like $e^{-Ct^{1/3}}$ for some constant $C > 0$. However, further estimates on the solution (such as, for example, uniform estimates on exponential moments) which are not readily available here are essential in [16] in order to show such a convergence. Finding the optimal rate of convergence for the DCF equations is an interesting open problem.

In the next section we present some preliminary definitions and known properties of the solutions. In Sect. 3 we prove the inequality which is used in Sect. 4 to identify the equilibrium to which a solution converges, and show our main result on the matter. In Sect. 5 we prove the inequality relating the relative energy to the free energy dissipation rate, and use it to give an explicit rate of convergence to equilibrium. Finally, in an Appendix we prove, under conditions suitable for our result, that the entropy functional is decreasing along solutions of the DCF equations. This is a result known under only slightly different conditions, and its proof is given in the appendix for completeness, though it does not contain essentially new arguments.

2 Preliminaries and Known Results

2.1 Definitions and Hypotheses

Let us first define precisely what we understand by a solution of the DCF equations (1):

Definition 2.1 A *solution* on the interval $[0, T)$ (for a given $T > 0$ or $T = \infty$) of (1) is a sequence of nonnegative functions $c_i : [0, T) \rightarrow [0, +\infty)$ ($i \geq 1$) such that

1. for all $i \geq 1$, c_i is absolutely continuous in compact sets of $[0, T)$ and $\sum_{i \geq 1} i c_i(t)$ is bounded on $[0, T)$,
2. for all $j \geq 1$, the sums $\sum_{i=1}^{\infty} a_{i,j} c_i(t)$ and $\sum_{i=1}^{\infty} b_{i,j} c_{i+j}(t)$ are finite for almost all $t \in [0, T)$,
3. and (1) hold for almost all $t \in [0, T)$.

If $\{c_i\}_{i \geq 1}$ is a solution to the DCF equations on some interval and t is in this interval, we will refer to the sum $\sum_{i \geq 1} i c_i(t)$ as its *mass at time t*. Actually, if $\{c_i\}_{i \geq 1}$ is any sequence of nonnegative numbers, we will call the sum $\sum_{i \geq 1} i c_i$ its mass. We say that a solution to the DCF equations *conserves the mass* when its mass at any time of its interval of definition is the same.

An *equilibrium* is a solution of the DCF equations which does not depend on time.

For the main results in this paper we will need some or all of the following hypotheses; note that Hypotheses 1–5 concern the coefficients $a_{i,j}, b_{i,j}$, while Hypothesis 6 concerns the initial data c^0 .

Hypothesis 1 (Growth of coefficients) *For integer $i, j \geq 1$, the coefficients $a_{i,j}$ and $b_{i,j}$ are nonnegative numbers, they are symmetric in i, j (this is, $a_{i,j} = a_{j,i}$ and $b_{i,j} = b_{j,i}$ for all $i, j \geq 1$) and for some constants $K > 0, 0 \leq \lambda < 1$ and $\gamma \in \mathbb{R}$,*

$$a_{i,j}, b_{i,j} \leq K(i^\lambda + j^\lambda) \quad \text{for all } i, j \geq 1, \tag{9}$$

$$\sum_{j=1}^{i-1} b_{j,i-j} \leq K i^\gamma \quad \text{for all } i \geq 1. \tag{10}$$

Hypothesis 2 (Detailed Balance) *There exists a positive sequence $\{Q_i\}_{i \geq 1}$ with $Q_1 = 1$ such that for all $i, j \geq 1$,*

$$a_{i,j} Q_i Q_j = b_{ij} Q_{i+j}. \tag{11}$$

Hypothesis 3 (Critical monomer concentration) *The sequence Q_i satisfies that:*

$$\lim_{j \rightarrow \infty} Q_j^{1/j} = \frac{1}{z_s} \quad \text{for some } 0 < z_s < \infty. \tag{12}$$

The critical density ρ_s is defined to be

$$\rho_s := \sum_{j=1}^{\infty} j Q_j z_s^j \quad (0 < \rho_s \leq \infty). \tag{13}$$

Hypothesis 4 (Regularity of Q_i) *The sequence $\{Q_i z_s^i\}_{i \geq 1}$ is decreasing.*

Hypothesis 5 (Strong coagulation of small particles) *For some constant $K_1 > 0$ it holds that*

$$a_{i,1} \geq K_1 i^\lambda \quad \text{for all } i \geq 1. \tag{14}$$

Hypothesis 6 (Moment of initial data) *The sequence $\{c_i^0\}_{i \geq 1}$ (which will be used as initial data later) is a sequence of nonnegative numbers with finite moments of orders $2 - \lambda$, $1 + \lambda$ and $1 + \gamma$; this is,*

$$\sum_{i \geq 1} i^\mu c_i^0 < +\infty \quad \text{for } \mu := \max\{2 - \lambda, 1 + \lambda, 1 + \gamma\}. \tag{15}$$

2.2 Existence of Solutions and Equilibria

Next we recall some known results on the existence, uniqueness and properties of solutions of the DCF equations.

In order to derive estimates on solutions it is often useful to see them as a limit of solutions of simpler systems. In this case it is common to consider the finite system of ordinary differential equations obtained by taking an $N \geq 1$ and writing the DCF equations with coefficients $a_{i,j}^N, b_{i,j}^N$, where

$$a_{i,j}^N = a_{i,j}, \quad b_{i,j}^N = b_{i,j} \quad \text{for } i + j \leq N, \tag{16}$$

$$a_{i,j}^N = b_{i,j}^N = 0 \quad \text{for } i + j > N, \tag{17}$$

and taking into account equations for c_i only up to $i = N$, while c_i are taken to be 0 for $i > N$. For any nonnegative initial data $\{c_i^0\}_{i \leq N}$ at $t = 0$ this finite system is shown to have a unique nonnegative solution defined on $[0, +\infty)$ [5].

Existence results are usually obtained by proving that the sequence of truncations just defined converges in some sense and its limit is a solution of the complete DCF equations. Let us state a result of this kind taken from Theorems 2.4 and 2.5 and Corollary 2.6 of [5]:

Proposition 2.2 (Existence of solutions) *Assume Hypothesis 1, and take any nonnegative sequence $\{c_i^0\}_{i \geq 1}$ with $\sum_{i \geq 1} i c_i^0 < +\infty$. Then there exists a mass-conserving solution c to the DCF equations (1) on $[0, +\infty)$ with $c(0) = c^0$.*

In addition, this solution is constructed as a limit of solutions of the truncated system defined at the beginning of Sect. 2.2 in the sense that, if $\{c_i^N\}_{i \leq N}$ is the solution of the finite truncated system with N equations and initial data $\{c_i^0\}_{i \leq N}$, then there is some sequence $\{N_k\}_k$ such that for all $T > 0$

$$\sup_{t \in (0, T)} \sum_{i \geq 1} i |c_i(t) - c_i^{N_k}(t)| \rightarrow 0 \quad \text{when } k \rightarrow \infty. \tag{18}$$

Note that c_i^N is taken to be 0 whenever $i > N$.

The following result on the existence of equilibria can be found in [11, Theorem 5.2]:

Proposition 2.3 Assume Hypotheses 1–3.

1. For $0 \leq \rho \leq \rho_s$ (or $\rho < +\infty$ if $\rho_s = +\infty$), there exists exactly one equilibrium $\{c_i^\rho\}_{i \geq 1}$ of (1) with density ρ , which is given by

$$c_i^\rho = Q_i z^i \quad \forall i \geq 1, \tag{19}$$

where $0 < z \leq z_s$ is the only positive number such that $\sum_{i=1}^\infty i Q_i z^i = \rho$.

2. For $\rho_s < \rho < +\infty$ there is no equilibrium of (1) with density ρ .

2.3 Lyapunov Functionals

Take $\{Q_i\}$ to be the sequence defined in Hypothesis 2; we assume Hypothesis 3 throughout. If $c = \{c_i\}$ is a nonnegative sequence with finite mass ($\sum_{i \geq 1} i c_i < +\infty$), then we define the free energy $V(c)$ as:

$$V(c) := \sum_{i=1}^\infty c_i \left(\log \frac{c_i}{Q_i} - 1 \right). \tag{20}$$

(Observe that Hypothesis 2 ensures that $Q_i > 0$ for all i .) When c is understood we will simply denote this as V . In [6, Lemma 4.2 and p. 680] it is proved that it is finite for all nonnegative c with finite mass and that, for any $\rho \geq 0$, it is bounded above and below on the set of nonnegative sequences $\{c_i\}$ such that $\sum_{i \geq 1} i c_i = \rho$ (always under Hypothesis 3; see also Lemmas 7.1 and 7.2).

If $0 < z \leq z_s$ and c is as above, we define the free energy relative to the equilibrium $\{Q_i z^i\}_{i \geq 1}$, or relative energy for short, by the following expression, after Jabin and Niethammer [16]:

$$F_z(c) := \sum_{i=1}^\infty c_i \left(\log \frac{c_i}{Q_i z^i} - 1 \right) + \sum_{i=1}^\infty Q_i z^i \tag{21}$$

$$= V(c) + \sum_{i=1}^\infty Q_i z^i - \log z \sum_{i \geq 1} i c_i \tag{22}$$

where c^{eq} represents the equilibrium $\{Q_i z^i\}_{i \geq 1}$. This is also clearly finite when $0 < z < z_s$; for $z = z_s$ it is finite when $\rho_s < +\infty$, but may be infinite when $\rho_s = +\infty$. Also, when the mass of c is finite and less than or equal to ρ_s , we can choose z so that the mass of the equilibrium $c^{eq} := \{Q_i z^i\}_{i \geq 1}$ is the same as that of c . In this case, $F_z(c)$ can be written as the difference between the free energy of c and that of the equilibrium with the same mass:

$$F_z(c) = V(c) - V(c^{eq}). \tag{23}$$

Finally, when $c_i > 0$ for all i , the free energy dissipation rate $D_{CF}(c)$ is defined as

$$D_{CF}(c) := \frac{1}{2} \sum_{i,j=1}^\infty a_{i,j} Q_i Q_j \left(\frac{c_i c_j}{Q_i Q_j} - \frac{c_{i+j}}{Q_{i+j}} \right) \left(\log \frac{c_i c_j}{Q_i Q_j} - \log \frac{c_{i+j}}{Q_{i+j}} \right) \geq 0. \tag{24}$$

Now, assume Hypotheses 1–3 and also that for all $i \geq 1$, $a_{i,1} > 0$ (which implies $b_{i,1} > 0$). Let $\{c_i\}_{i \geq 1}$ be the solution to the DCF equations (1) on $[0, +\infty)$ given by Proposition 2.2 under these hypotheses. The positivity assumption on $a_{i,1}$ above ensures that $c_i(t) > 0$ for

all $t > 0$ and $i \geq 1$ (see [10] or [11, Theorem 5.2]), so $D_{CF}(c(t))$ makes sense for $t > 0$. Denote $V \equiv V(c(t))$, $F_z \equiv F_z(c(t))$ and $D_{CF} \equiv D_{CF}(c(t))$ for $t \geq 0$. We prove in Sect. 6 that if the initial condition has certain finite moments, then both V and F_z are absolutely continuous on compact sets and

$$\frac{d}{dt} F_z = \frac{d}{dt} V = -D_{CF} \quad \text{for almost all } t > 0. \tag{25}$$

Then, V and F_z are decreasing along mass-conserving solutions of (1), so they are Lyapunov functionals for this equation (they differ by a constant along a given solution). We will be especially interested in studying F_z , as it is positive and measures the proximity of a solution to the equilibrium $\{Q_i z^i\}$ in the following sense, taken from [16, Lemma 3.1]: if $c = \{c_i\}_{i \geq 1}$ is a nonnegative sequence with finite mass and $0 < z < z_s$, then there is a constant K_z such that

$$\sum_{i \geq 1} i |c_i - Q_i z^i| \leq \max\{2F_z(c), K_z \sqrt{F_z(c)}\}. \tag{26}$$

The constant K_z can be taken to be

$$K_z := \frac{1}{1 - \sqrt{\mu}} - 1 \quad \text{with } \mu := \frac{z}{z_s}. \tag{27}$$

A consequence of this is that, for a given mass-conserving solution c of (1) with mass ρ , $F_z(c(t)) \rightarrow 0$ as $t \rightarrow +\infty$ for some $0 < z < z_s$ implies that z is such that $\sum_{i \geq 1} i Q_i z^i = \rho$.

Below we will use the free energy dissipation rate which appears in the Becker–Döring equations: for a strictly positive sequence $\{c_i\}_{i \geq 1}$ we set

$$D := \sum_{i=1}^{\infty} a_i Q_i \left(\frac{c_1 c_i}{Q_i} - \frac{c_{i+1}}{Q_{i+1}} \right) \left(\log \frac{c_1 c_i}{Q_i} - \log \frac{c_{i+1}}{Q_{i+1}} \right) \tag{28}$$

where

$$a_1 := \frac{1}{2} a_{1,1}, \tag{29}$$

$$a_i := a_{i,1} \quad \text{for } i > 1. \tag{30}$$

Note that $0 \leq D \leq D_{CF}$, as every term in D already appears in D_{CF} , taking into account the symmetry of $a_{i,j}$.

2.4 H-Theorem

As was already pointed out above, formally one can calculate the time derivative of the free energy to obtain that

$$\frac{d}{dt} V = -D_{CF}. \tag{31}$$

This result has been proved rigorously in [11, Theorem 5.2] under a growth hypothesis on the coefficients $a_{i,j}$, $b_{i,j}$ and some further regularity assumptions given as conditions on the sequence Q_i ; for the continuous equations, the corresponding result was proved in [17] by assuming a stronger regularity of the initial condition (namely, the boundedness of certain

moments) and comparatively weaker regularity of the coefficients $a_{i,j}, b_{i,j}$. Here we would like to prove the result for the discrete equations in a way similar to that in [11], but which uses hypotheses analogous to those in [17]. This result is more natural in our context, as anyway an essential point of the proof of our main result relies on moment estimates for the solution.

Theorem 2.4 *Assume Hypotheses 1–3 and also that*

1. *the initial data $c^0 = \{c_i^0\}_{i \geq 1}$ is nonnegative and has finite moments of order $1 + \lambda$ and $1 + \gamma$,*
2. *$a_{1,i}, b_{1,i} > 0$ for all $i \geq 1$.*

(We recall that λ and γ are defined in Hypothesis 1.) Let c be the solution of the DCF equations given by Theorem 2.2. Then, D_{CF} is locally integrable and

$$\frac{d}{dt}V = -D_{CF} \quad \text{for almost all } t \geq 0. \tag{32}$$

The proof of this is given in Sect. 6, as it is only a slight variation of well-known proofs such as those in [11, 17].

2.5 Weak Convergence of Solutions to an Equilibrium

As stated in Theorem 2.4, the free energy V (defined in (20)) is decreasing along solutions of the DCF equations. It is known that this implies that every solution must converge in a weak sense to a certain equilibrium with mass less than or equal to that of the solution itself, as is shown for example in [11, Theorem 6.4]. We state this in the following result, which is known to hold under slightly different hypotheses; its proof follows from the H-theorem 2.4 in the same way as in [11, Theorem 6.4]:

Proposition 2.5 *Assume the same hypotheses as in Theorem 2.4. Let $c = \{c_j\}$ be a solution of (1) on $[0, \infty)$ which conserves mass, and call its mass ρ .*

Then there exists $0 \leq z \leq z_s$ such that $\sum_{i \geq 1} i Q_i z^i \leq \rho$ and

$$\lim_{t \rightarrow +\infty} c_i(t) = Q_i z^i \quad \text{for all } i \geq 1. \tag{33}$$

The above convergence is usually referred to as *weak-** convergence. Precisely, we say that a sequence $c^n = \{c_i^n\}_{i \geq 1}$ converges weak-* to a sequence $c = \{c_i\}_{i \geq 1}$ if

- $\sum_{i \geq 1} i |c_i^n| \leq K$ for some $K > 0$ and all $n \geq 1$, and
- for each $i \geq 1$, $c_i^n \rightarrow c_i$ as $n \rightarrow \infty$.

We denote this as $c^n \xrightarrow{*} c$. There is also a useful relationship between weak-* and strong convergence:

Lemma 2.6 ([6], Lemma 3.3) *If $\{c^n\}$ is a sequence such that $c^n \xrightarrow{*} c$ and $\sum_{i \geq 1} i |c_i^n| \rightarrow \sum_{i \geq 1} i |c_i|$, then $\sum_{i \geq 1} i |c_i^n - c_i| \rightarrow 0$.*

3 Mass Difference Estimate

In this section we prove the following inequality, which is the fundamental result needed to prove our main result, Theorem 4.1:

Proposition 3.1 *Assume Hypotheses 1–5 and take a strictly positive sequence $c = \{c_i\}_{i \geq 1}$. Suppose that*

- $0 < c_1 < z_s$ and
- $M_{2-\lambda} := \sum_{i \geq 1} i^{2-\lambda} c_i < +\infty$,

and call $D \equiv D(c)$ the Becker–Döring free energy dissipation rate from (28). Then,

$$\sum_{i \geq 1} i c_i - \sum_{i \geq 1} i Q_i c_1^i \leq C \sqrt{D} \sqrt{M_{2-\lambda}} \tag{34}$$

for some constant C depending only on the coefficients $a_{i,j}$, $b_{i,j}$ and increasingly on the quantity c_1/z_s .

The lemma which follows will be used in the proof of this result:

Lemma 3.2

$$\sum_{i=j+1}^{\infty} i Q_i c_1^i \leq C j Q_{j+1} c_1^{j+1} \quad \text{for all } j \geq 1, \tag{35}$$

where C can be taken to be

$$C = 3 \frac{z_s^2}{(z_s - c_1)^2}. \tag{36}$$

Proof Using the hypothesis that $Q_i z_s^i$ is decreasing in i , and calling $r := c_1/z_s$,

$$\begin{aligned} \sum_{i=j+1}^{\infty} i Q_i c_1^i &\leq Q_{j+1} z_s^{j+1} \sum_{i=j+1}^{\infty} i \left(\frac{c_1}{z_s}\right)^i \\ &= Q_{j+1} z_s^{j+1} \left(\frac{c_1}{z_s}\right)^{j+1} \sum_{i'=0}^{\infty} (i' + j + 1) \left(\frac{c_1}{z_s}\right)^{i'} \\ &= Q_{j+1} c_1^{j+1} \left(\frac{r}{(1-r)^2} + (j + 1) \frac{1}{1-r}\right) \\ &\leq 3 \frac{1}{(1-r)^2} j Q_{j+1} c_1^{j+1} = C j Q_{j+1} c_1^{j+1}, \end{aligned} \tag{37}$$

where the last inequality is obtained by observing that both $\frac{r}{(1-r)^2}$ and $\frac{1}{1-r}$ are smaller than $\frac{1}{(1-r)^2}$, and that $j + 2 \leq 3j$ for all $j \geq 1$. □

Proof of Proposition 3.1 Call

$$u_i := \frac{c_i}{Q_i c_1^i}. \tag{38}$$

With this notation, D can be rewritten as

$$D = \sum_{i=1}^{\infty} a_i Q_i c_1^{i+1} (u_i - u_{i+1}) (\log u_i - \log u_{i+1}). \tag{39}$$

Noting that $u_1 = 1$ and using Lemma 3.2 we can write

$$\begin{aligned} & \sum_{i \geq 1} i c_i - \sum_{i \geq 1} i Q_i c_1^i \\ &= \sum_{i \geq 1} i Q_i c_1^i (u_i - 1) \\ &= \sum_{i \geq 1} i Q_i c_1^i \sum_{j=1}^{i-1} (u_{j+1} - u_j) = \sum_{j=1}^{\infty} (u_{j+1} - u_j) \sum_{i=j+1}^{\infty} i Q_i c_1^i \\ &\leq \sum_{\substack{u_{j+1} \geq u_j \\ j \geq 1}} (u_{j+1} - u_j) \sum_{i=j+1}^{\infty} i Q_i c_1^i \leq C_1 \sum_{\substack{u_{j+1} \geq u_j \\ j \geq 1}} j Q_{j+1} c_1^{j+1} (u_{j+1} - u_j) \\ &\leq \frac{C_1}{\sqrt{z_s}} \left(\sum_{\substack{u_{j+1} \geq u_j \\ j \geq 1}} a_j Q_j c_1^{j+1} \frac{(u_{j+1} - u_j)^2}{u_{j+1}} \right)^{1/2} \left(\sum_{j \geq 1} \frac{j^2}{a_j} Q_{j+1} c_1^{j+1} u_{j+1} \right)^{1/2}, \tag{40} \end{aligned}$$

where C_1 is the constant in Lemma 3.2. Here we have used the Cauchy–Schwarz inequality and the fact that $z_s Q_{j+1} \leq Q_j$ for all j (Hypothesis 4). Now, with

$$\frac{(x - y)^2}{\max\{x, y\}} \leq (x - y) (\log x - \log y) \quad \text{for all } x, y > 0, \tag{41}$$

one sees from (39) that the first parenthesis is less than D . For the second one, use Hypothesis 5 to write

$$\sum_{j \geq 1} \frac{j^2}{a_j} Q_{j+1} c_1^{j+1} u_{j+1} \leq \frac{1}{K_1} \sum_{j \geq 1} (j + 1)^{2-\lambda} c_{j+1} \leq \frac{1}{K_1} M_{2-\lambda}. \tag{42}$$

We finally obtain that

$$\sum_{i \geq 1} i c_i - \sum_{i \geq 1} i Q_i c_1^i \leq C \sqrt{D} \sqrt{M_{2-\lambda}}, \tag{43}$$

with $C := \frac{C_1}{\sqrt{z_s} \sqrt{K_1}}$ (we recall that C_1 is the constant in Lemma 3.2 and that K_1 is defined in Hypothesis 5; note that C_1 is a constant with the dependence described in Proposition 3.1). This proves the proposition. \square

4 Strong Convergence to Equilibrium

Let $c = \{c_i\}_{i \geq 1}$ be the solution to the DCF equations (1) on $[0, +\infty)$ with initial data $c^0 = \{c_i^0\}_{i \geq 1}$ given by Proposition 2.2 under Hypotheses 1–6. Below we will always denote by ρ

the mass of the solution c , which is constant:

$$\rho := \sum_{i \geq 1} i c_i(t) \quad \text{for any } t \geq 0. \tag{44}$$

Of course, if $\rho = 0$ then the solution itself is constantly 0 and is uninteresting, so we will assume that $\rho > 0$. Our main result, Theorem 1.1, is stated more precisely as follows:

Theorem 4.1 *Assume the Hypotheses 1–6. If $\rho > \rho_s$, then*

$$c_i(t) \rightarrow Q_i z_s^i \quad \text{for all } i \geq 1, \tag{45}$$

while if $\rho \leq \rho_s$, then c converges strongly to the only equilibrium with mass ρ :

$$\sum_{i \geq 1} i |c_i(t) - Q_i z^i| \rightarrow 0 \quad \text{when } t \rightarrow +\infty \tag{46}$$

for the only $z \geq 0$ such that

$$\rho = \sum_{i \geq 1} i Q_i z^i. \tag{47}$$

By well-known arguments (see for example [6, 8, 11]) this theorem follows from Proposition 2.5 if we can show that whenever a solution converges weak-* to an equilibrium of mass strictly below ρ_s , then the convergence must also be strong; the latter result will be proved below in Proposition 4.2. The reason that this is enough is the following: by Proposition 2.5, we know that every solution must converge, at least weak-*, to some equilibrium with its same mass or less. Then, if one has Proposition 4.2, one obviously has Theorem 4.1 for any solution with mass $\rho < \rho_s$. For a solution with mass $\rho = \rho_s$, the weak-* limit must be the equilibrium with mass ρ_s , as any other limit with mass strictly less than ρ_s implies strong convergence by Proposition 4.2, which is absurd (a strong limit must have the same mass as the solution which converges to it). By Lemma 2.6, the convergence to the equilibrium with mass ρ_s must be strong, as both masses coincide. This proves Theorem 4.1 for a solution with mass $\rho = \rho_s$. Finally, for a solution with mass $\rho > \rho_s$, the same argument shows that its only possible weak-* limit is the equilibrium with mass ρ_s , which completes the statement of Theorem 4.1. For further detail on this, the reader can look at the references mentioned above ([6, 8, 11]).

Proposition 4.2 *If the solution c converges weak-* to an equilibrium $\{Q_i z^i\}$ with $0 \leq z < z_s$, then $\rho < \rho_s$, and z is the only number such that*

$$\rho = \sum_{i \geq 1} i Q_i z^i := \rho_z \tag{48}$$

and the convergence is strong, in the sense that

$$\sum_{i \geq 1} i |c_i(t) - Q_i z^i| \rightarrow 0 \quad \text{when } t \rightarrow +\infty. \tag{49}$$

The bound in the previous section will be the fundamental tool to prove the above proposition. In addition, we will need the following two lemmas: the first one is a simple inequality

which has been often used in this context (see [9, Appendix D] for a discussion of this inequality and related ones), and which we prove for completeness. It will be used to prove a bound on the increase of the moment of order $2 - \lambda$ of a solution, given in Lemma 4.5.

Lemma 4.3 *For $0 \leq \lambda \leq 1$ and $1 \leq k \leq 2 - \lambda$ there is a constant $C_{k,\lambda} \geq 0$ such that*

$$(x^\lambda + y^\lambda)((x + y)^k - x^k - y^k) \leq C_{k,\lambda}(xy)^{\frac{\lambda+k}{2}} \quad \text{for all } x, y \geq 0. \tag{50}$$

Proof If any of x or y is zero, the inequality is trivial, so take $x, y > 0$. By symmetry, it is clearly enough to prove it when $x \leq y$. To do this, call $r := x/y$, so that $0 < r \leq 1$. We have

$$\begin{aligned} (1 + r^\lambda)((1 + r)^k - 1 - r^k) &\leq (1 + 2^\lambda)(kr(1 + r)^{k-1} - r^k) \\ &\leq (1 + 2^\lambda)2^{k-1}kr \leq (1 + 2^\lambda)2^{k-1}kr^{\frac{\lambda+k}{2}}. \end{aligned} \tag{51}$$

In the first inequality we have used the mean value theorem and $k \geq 0$; in the second one we have left out the negative term and used that $k \geq 1$ and $r \leq 1$; and for the third one we have used that $1 \geq (\lambda + k)/2$ and $0 < r \leq 1$. Now, multiplying the beginning and end of the previous inequality by $y^{\frac{\lambda+k}{2}}$ and recalling the definition of r gives the inequality of the lemma. \square

Remark 4.4 Note that the inequality is also true, but of no value, for $k < 1$, as then the part on the left is negative and that on the right is positive. Also, note that in the previous lemma the constant can be chosen to be independent of k, λ .

Lemma 4.5 *Under the hypotheses of Proposition 4.2, there is some constant $C > 0$ which depends only on $M_{2-\lambda}(0), \rho,$ and the constant K in Hypothesis 1 such that*

$$M_{2-\lambda}(t) := \sum_{i=1}^{\infty} i^{2-\lambda} c_i(t) \leq C(1 + t) \quad \text{for all } t \geq 0. \tag{52}$$

Proof Take c^N to be the solution of the finite system of size N from the beginning of Sect. 2.2 with initial data $\{c_i^0\}_{i \leq N}$, and set $c_i^N := 0$ for $i > N$. We will prove the estimate for any such solution and a constant C depending only on the quantities in the lemma (and hence independent of N), and then a usual argument [5, 8] allows us to pass to the limit and get the same bound for the complete solution c . In fact, we will denote c^N as c to simplify the notation. Using a well-known identity giving the time derivative of moments of the solution c [8] we have,

$$\begin{aligned} \frac{d}{dt} M_{2-\lambda}(t) &\leq \frac{1}{2} \sum_{i,j=1}^{\infty} a_{i,j} c_i(t) c_j(t) ((i + j)^{2-\lambda} - i^{2-\lambda} - j^{2-\lambda}) \\ &\leq K C' \sum_{i,j=1}^{\infty} c_i(t) c_j(t) i j = K C' \rho^2, \end{aligned} \tag{53}$$

where K is the constant in Hypothesis 1 and C' is the one in Lemma 4.3 for $k = 2 - \lambda$. Then, for all $t \geq t_0$,

$$M_{2-\lambda}(t) \leq M_{2-\lambda}(0) + K C' \rho^2 t \tag{54}$$

which proves the lemma with, for example, $C := M_{2-\lambda}(0) + K C' \rho^2$. \square

Now, let us prove Proposition 4.2:

Proof of Proposition 4.2 It is enough to prove the first statement (that $\rho = \rho_z$, the mass of the equilibrium to which the solution converges weak-*,) as then the strong convergence follows from Lemma 2.6. Note that we already know that $\rho \geq \rho_z$ thanks to Proposition 2.5—only loss of mass, not gain, can take place in the large time limit—so we only need to prove that $\rho \leq \rho_z$.

As the solution c converges weak-* to $\{Q_i z^i\}$, we know that $c_1 \rightarrow z < z_s$ and after some time $t_0 > 0$ it holds that

$$c_1(t) \leq \frac{z + z_s}{2} < z_s \quad \text{for all } t \geq t_0. \tag{55}$$

Then, calling $\rho_1(t) := \sum_{i \geq 1} i Q_i c_1^i(t)$ and applying Proposition 3.1 to $c(t)$ for $t \geq t_0$ we have, for some fixed constants C_1, C_2 ,

$$\rho - \rho_1(t) \leq C_1 \sqrt{D} \sqrt{M_{2-\lambda}(t)} \leq C_2 \sqrt{D} \sqrt{1+t} \quad \text{for } t \geq t_0, \tag{56}$$

thanks to Lemma 4.5. Now we note that

$$\lim_{t \rightarrow +\infty} \rho_1(t) = \rho_z = \sum_{i \geq 1} i Q_i z^i, \tag{57}$$

which is a consequence of the continuity in z of the above power series (which has radius of convergence z_s) and the fact that $c_1(t) \rightarrow z$ as $t \rightarrow \infty$.

We would like to obtain a lower bound for D from (56), but this can only be done when the left hand side is positive. Let us see that we can suppose this to hold after a certain time t_1 : otherwise there is a sequence $t_n \rightarrow \infty$ such that $\rho - \rho_1(t_n) \leq 0$, or $\rho \leq \rho_1(t_n) \rightarrow \rho_z$, so $\rho \leq \rho_z$ and the statement is proved. So we can assume that there is a time $t_1 \geq t_0$ such that $\rho > \rho_1(t)$ for all $t \geq t_1$.

Then, for all $t \geq t_1$, (56) implies that

$$D(t) \geq C_3 \frac{(\rho - \rho_1(t))^2}{1+t} \quad \text{for } t \geq t_1, \tag{58}$$

for $C_3 := 1/C_2^2$. Now, if V represents the free energy of the solution c , we know that for $t \geq t_1$

$$V(t) = V(t_1) - \int_{t_1}^t D_{CF}(s) ds \tag{59}$$

$$\leq V(t_1) - \int_{t_1}^t D(s) ds \tag{60}$$

$$\leq V(t_1) - C_3 \int_{t_1}^t \frac{(\rho - \rho_1(s))^2}{1+s} ds. \tag{61}$$

As V is bounded below for all times, we see the right hand side must be bounded for all times $t \geq t_1$; hence, knowing from (57) that $\rho_1(t)$ has a limit as $t \rightarrow \infty$, this proves that its limit is ρ . On the other hand, its limit is ρ_z according to (57), so it must be $\rho = \rho_z$, which finishes the proof. \square

5 Relative Energy Estimate

Take a nonnegative sequence $c = \{c_i\}_{i \geq 1}$ with $0 < c_1 < z_s$. We are interested in estimating the relative energy F_{c_1} of c to $\{Q_i c_1^i\}_{i \geq 1}$, a strategy also used in [16]. For brevity, we denote $F \equiv F_{c_1}(c)$ and write

$$u_i := \frac{c_i}{Q_i c_1^i}, \tag{62}$$

so that F can be rewritten as

$$F := \sum_{i=1}^{\infty} c_i \left(\log \frac{c_i}{Q_i c_1^i} - 1 \right) + \sum_{i=1}^{\infty} Q_i c_1^i \tag{63}$$

$$= \sum_{i=1}^{\infty} Q_i c_1^i (u_i \log u_i - u_i + 1) \tag{64}$$

$$= \sum_{i=1}^{\infty} Q_i c_1^i f(u_i), \tag{65}$$

where

$$f(x) := x \log x - x + 1 \quad \text{for } x > 0. \tag{66}$$

Note that F is finite if $0 < c_1 < z_s$.

With the same notation $D = D(c)$ can be rewritten as in (39), which we recall here:

$$D = \sum_{i=1}^{\infty} a_i Q_i c_1^{i+1} (u_i - u_{i+1}) (\log u_i - \log u_{i+1}). \tag{67}$$

In this section we show the following result:

Proposition 5.1 *Assume Hypotheses 1–6, and let $c = \{c_i\}_{i \geq 1}$ be a strictly positive sequence with $0 < c_1 < z_s$ and with $M_{2-\lambda} := \sum_{i \geq 1} i^{2-\lambda} c_i < +\infty$. Then there is some constant $C \geq 0$ that depends only on the coefficients $a_{i,j}, b_{i,j}$ ($i, j \geq 1$), on ρ and continuously on c_1 such that*

$$F \leq C \max\{\sqrt{D} \sqrt{M_{2-\lambda}}, D\} \tag{68}$$

where $D = D(c)$ is the Becker–Döring free energy dissipation term defined in (28).

Remark 5.2 The constant C in the previous proposition may become infinite as c_1 approaches 0 or z_s ; however, we specify that it depends *continuously* on c_1 so that, if one knows that $\epsilon < c_1 < z_s - \epsilon$ for some $\epsilon > 0$, then the constant may be taken to depend on ϵ and not on c_1 . This will be used in the proof of Proposition 6.1.

Remark 5.3 The dependence on c_1 of the above inequality may be of interest; for example, if one wants to use it to prove Theorem 4.1 (instead of the inequality in Proposition 3.1), one needs some control on the constant as $c_1 \rightarrow 0$ in order to rule out the possibility that solutions converge weakly to the equilibrium with mass 0 (i.e., $c_i \equiv 0$ for all i). We have not explicitly stated this dependence for simplicity (as it is not used, makes the proof somewhat more cumbersome, and after Theorem 4.1 we know that c_1 is greater than some positive constant after a certain time anyway), but the reader can check from the constants in the proof that the growth of C as $c_1 \rightarrow 0$ is controlled by $|\log c_1|$.

Let us prove the above inequality. Of course, the inequality is nontrivial only when $D < +\infty$, so we assume that D is finite. The case $D = 0$ is also trivial, for if D vanishes, c must be a nonzero equilibrium, and then $F = 0$; hence, we will also assume that $D > 0$.

In the course of the present proof the letters C, C_1, C_2, \dots will always be used to denote numbers which depend on the quantities allowed in the statement of Proposition 5.1, and in the way specified there. For short, we will frequently refer to these as “allowed constants”.

Take any integer $N \geq 1$ and split the sum in F as

$$F = \sum_{i \leq N} Q_i c_1^i f(u_i) + \sum_{i > N} Q_i c_1^i f(u_i) =: F_1 + F_2. \tag{69}$$

First step: estimate for F_1 . As $u_1 = 1$ and $f(1) = 0$,

$$f(u_i) = \sum_{j=1}^{i-1} (f(u_{j+1}) - f(u_j)) \quad \text{for } i \geq 1. \tag{70}$$

Note that the sum is empty for $i = 1$. With this,

$$\begin{aligned} F_1 &= \sum_{i=1}^N Q_i c_1^i f(u_i) = \sum_{i=1}^N \sum_{j=1}^{i-1} Q_i c_1^i (f(u_{j+1}) - f(u_j)) \\ &= \sum_{j=1}^{N-1} (f(u_{j+1}) - f(u_j)) \sum_{i=j+1}^N Q_i c_1^i \\ &\leq C_1 \sum_{\substack{j=1 \\ f(u_{j+1}) \geq f(u_j)}}^N Q_{j+1} c_1^{j+1} (f(u_{j+1}) - f(u_j)), \end{aligned} \tag{71}$$

where the last inequality, for some allowed constant C_1 , is obtained in a very similar way to that in Lemma 3.2.

Lemma 5.4 For $x, y > 0$ it holds that

$$f(x) - f(y) \leq (x - y)(\log x - \log y) + (x - y) \log \max\{x, y\}. \tag{72}$$

Proof Regardless of the sign of $x - y$, the mean value theorem shows that

$$\begin{aligned} f(x) - f(y) &\leq (x - y) \log x \\ &= (x - y)(\log x - \log y) + (x - y) \log y \\ &\leq (x - y)(\log x - \log y) + (x - y) \log \max\{x, y\}. \end{aligned} \tag{73}$$

Again, notice that the last step holds both when $x \leq y$ and $y \leq x$. □

With the previous lemma we can continue from (71). Denoting $w_j := \max\{u_j, u_{j+1}\}$,

$$\begin{aligned}
 F_1 &\leq C_1 \sum_{\substack{j=1 \\ f(u_{j+1}) \geq f(u_j)}}^N (f(u_{j+1}) - f(u_j)) Q_{j+1} c_1^{j+1} \\
 &\leq C_1 \sum_{j \leq N} Q_{j+1} c_1^{j+1} (u_{j+1} - u_j) (\log u_{j+1} - \log u_j) \\
 &\quad + C_1 \sum_{j \leq N} Q_{j+1} c_1^{j+1} |u_{j+1} - u_j| |\log w_j| =: T_1 + T_2.
 \end{aligned} \tag{74}$$

For the first term, T_1 , we can use once more that $z_s Q_{i+1} \leq Q_i$ (Hypothesis 4), the lower bound on a_j from Hypothesis 5 and the expression of D in (67) to see that

$$T_1 \leq 2 \frac{C_1}{K_1 z_s} D =: C_2 D, \tag{75}$$

where K_1 is the constant in Hypothesis 5 (and the factor of 2 appears because of the definition of a_1 in (29)). For the second term in (74), T_2 , the Cauchy–Schwarz inequality gives

$$T_2 \leq C_3 \left(\sum_{j \leq N} a_j Q_j c_1^{j+1} \frac{(u_{j+1} - u_j)^2}{w_j} \right)^{1/2} \left(\sum_{j=1}^N \frac{1}{a_j} Q_{j+1} c_1^{j+1} w_j (\log w_j)^2 \right)^{1/2}, \tag{76}$$

where $C_3 := \frac{C_1}{\sqrt{z_s}}$, again using that $z_s Q_{i+1} \leq Q_i$ (Hypothesis 4). By inequality (41) and (67), the first term inside parentheses is less than D , so

$$T_2 \leq C_3 \sqrt{D} \left(\sum_{j=1}^N \frac{1}{a_j} Q_{j+1} c_1^{j+1} w_j (\log w_j)^2 \right)^{1/2}. \tag{77}$$

Now let us use the following result to compare $w_j (\log w_j)^2$ with $f(w_j)$:

Lemma 5.5 *It holds that*

$$x (\log x)^2 \leq 4(x \log x - x + 1) \max\{1, \log x\} \quad \text{for } x > 0. \tag{78}$$

Proof Call $g(x) := x (\log x)^2$ and $f(x) := (x \log x - x + 1)$ as before. Then, $f(1) = f'(1) = g(1) = g'(1) = 0$, $f''(x) = 1/x$ and

$$g''(x) = 2 \frac{\log x}{x} + 2 \frac{1}{x} \leq 4 f''(x) \quad \text{for } 0 < x \leq e, \tag{79}$$

so by integrating one gets $g(x) \leq 4f(x)$ for $0 < x \leq e$ and we have proved the inequality in this range.

Now, for $x \geq e$, we have $\log x \geq 1$ and the inequality is equivalent to showing that

$$3x \log x - 4(x - 1) \geq 0 \quad \text{for } x \geq e, \tag{80}$$

but the derivative of this function is $3 \log x - 1$, which is clearly positive for $x \geq e$; hence, the function itself is greater than its value at $x = e$, which is $3e - 4(e - 1) = 4 - e > 0$. This finishes the proof. \square

With the previous lemma,

$$\frac{1}{a_j} w_j (\log w_j)^2 \leq \frac{4}{a_j} f(w_j) \max\{1, \log w_j\} \leq 4M_N f(w_j) \quad \text{for } j \leq N, \tag{81}$$

where M_N is the maximum for $j \leq N$ of the expression $\frac{1}{a_j} \max\{1, \log w_j\}$. Hence, continuing from (77),

$$T_2 \leq 2C_3 \sqrt{D} \sqrt{M_N} \left(\sum_{j=1}^N Q_{j+1} c_1^{j+1} f(w_j) \right)^{1/2}. \tag{82}$$

Now note that

$$f(w_j) \leq f(u_j) + f(u_{j+1}) \tag{83}$$

and again that $z_s Q_{j+1} \leq Q_j$ (Hypothesis 4) to get

$$\sum_{j=1}^N Q_{j+1} c_1^{j+1} f(w_j) \leq \frac{c_1}{z_s} \sum_{j=1}^N Q_j c_1^j f(u_j) + \sum_{j=1}^N Q_{j+1} c_1^{j+1} f(u_{j+1}) \leq C_4 F, \tag{84}$$

where C_4 can be taken to be $1 + c_1/z_s$, an allowed constant. Hence, from (82),

$$T_2 \leq C_5 \sqrt{D} \sqrt{M_N} \sqrt{F}, \tag{85}$$

with $C_5 := 2C_3 \sqrt{C_4}$. Observe that, as $c_i \leq \rho/i$ for $i \geq 1$,

$$\log u_i \leq \log \frac{\rho}{i Q_i c_1^i} = i \left(\log \frac{\rho^{1/i}}{i^{1/i} Q_i^{1/i}} + \log \frac{1}{c_1} \right) \leq C_7 i \tag{86}$$

for some allowed constant C_7 . We have used that $Q_i^{1/i}$ is bounded below by some constant thanks to Hypothesis 3, and thus the term inside the parentheses is bounded above by some allowed constant. Knowing that $a_i \geq K_1 i^\lambda$ (Hypothesis 5),

$$M_N \leq 2C_7 \frac{1}{K_1} N^{1-\lambda} =: C_8 N^{1-\lambda}, \tag{87}$$

and from (85),

$$T_2 \leq C_9 \sqrt{D} \sqrt{N^{1-\lambda}} \sqrt{F}, \tag{88}$$

with $C_9 := C_5 \sqrt{C_8}$. Now, putting together (74), (75) and (88) we have

$$\begin{aligned} F_1 &\leq C_2 D + C_9 \sqrt{N^{1-\lambda} D} \sqrt{F} \\ &\leq C_2 D + \frac{1}{2} F + \frac{C_9^2}{2} N^{1-\lambda} D \\ &\leq \frac{1}{2} F + C_{10} N^{1-\lambda} D, \end{aligned} \tag{89}$$

with $C_{10} := C_2 + C_9^2/2$.

Second step: Estimate for F_2 . (In this step, the symbols C_1, C_2, \dots are used again for convenience to denote allowed constants, but they have nothing to do with previous appearances of them.) We have

$$\begin{aligned}
 F_2 &= \sum_{i>N} Q_i c_1^i u_i \log u_i - \sum_{i>N} c_i + \sum_{i>N} Q_i c_1^i \\
 &\leq \sum_{i>N} Q_i c_1^i u_i \log u_i + \frac{1}{N} \sum_{i>N} i Q_i c_1^i \tag{90}
 \end{aligned}$$

$$\leq \sum_{i>N} Q_i c_1^i u_i \log u_i + \frac{1}{N} C_1, \tag{91}$$

where C_1 is $\sum_{i \geq 1} i Q_i c_1^i$, an allowed constant. For the other term one has, writing $\Psi(x) := x \log x$ (a superadditive function) and taking some constant $1 \geq C_2 > 0$ such that $Q_i^{1/i} \geq C_2$ (which is possible by Hypothesis 3),

$$\sum_{i>N} Q_i c_1^i u_i \log u_i = \sum_{i>N} (Q_i c_1^i u_i) \log(Q_i c_1^i u_i) - \sum_{i>N} Q_i c_1^i u_i \log(Q_i c_1^i) \tag{92}$$

$$\leq \Psi\left(\sum_{i>N} c_i\right) + \log \frac{C_2}{c_1} \sum_{i>N} i c_i. \tag{93}$$

Now, take $N \geq \rho$, so that $\sum_{i>N} c_i \leq \rho/N \leq 1$, which makes the first term negative. Then, calling $C_3 := \log \frac{C_2}{c_1}$ and continuing from above,

$$\sum_{i>N} Q_i c_1^i u_i \log u_i \leq \frac{C_3}{N^{1-\lambda}} \sum_{i>N} i^{2-\lambda} c_i \leq \frac{C_3}{N^{1-\lambda}} M_{2-\lambda}. \tag{94}$$

Together with (91) we obtain

$$F_2 \leq \frac{C_3}{N^{1-\lambda}} M_{2-\lambda} + \frac{C_1}{N} \leq \frac{C_4}{N^{1-\lambda}} M_{2-\lambda}, \tag{95}$$

with $C_4 := C_1 + C_3$.

Third step: Estimate for F . Again in this step, constants C_1, C_2, \dots have nothing to do with previous ones unless explicitly noted. With (89) and (95) we have, for any $N \geq \rho$,

$$F \leq \frac{1}{2} F + C_{10} N^{1-\lambda} D + \frac{C_4}{N^{1-\lambda}} M_{2-\lambda}, \tag{96}$$

where C_{10} is the constant from (89) and C_4 is that from (95). Hence, taking $C := \max\{2C_{10}, 2C_4\}$,

$$F \leq C \left(N^{1-\lambda} D + \frac{1}{N^{1-\lambda}} M_{2-\lambda} \right) \quad \text{for all integers } N \geq \rho. \tag{97}$$

Actually, it is clear that if we take $C_1 := 2C$ one can write the above for all real N such that $R := N^{1-\lambda} \geq (\rho + 2)^{1-\lambda} =: C_2$ instead of only for the integers, just by applying the previous inequality to the integer closest to N :

$$F \leq C_1 \left(R D + \frac{1}{R} M_{2-\lambda} \right) \quad \text{for all real } R \text{ with } R \geq C_2. \tag{98}$$

Let us choose R in a way that gives a suitable inequality:

- If $\frac{\sqrt{M_{2-\lambda}}}{\sqrt{D}} \geq C_2$, then we take $R := \frac{\sqrt{M_{2-\lambda}}}{\sqrt{D}}$ and we obtain

$$F \leq C_1 \sqrt{D} \sqrt{M_{2-\lambda}}. \tag{99}$$

- Otherwise, if $\frac{\sqrt{M_{2-\lambda}}}{\sqrt{D}} < C_2$ then $M_{2-\lambda} \leq C_2^2 D$ and inequality (98) with $R := C_2$ gives

$$F \leq C_1 C_2 D + \frac{C_1}{C_2} M_{2-\lambda} \leq C_1 C_2 D + \frac{C_1}{C_2} C_2^2 D = C_3 D, \tag{100}$$

with $C_3 := 2C_1 C_2$.

Equations (99) and (100) prove that, for $C := \max\{C_1, C_3\}$,

$$F \leq C \max\{\sqrt{D} \sqrt{M_{2-\lambda}}, D\}, \tag{101}$$

which proves the result.

6 Rate of Convergence to Equilibrium

With the previous results one can easily obtain the following rate of convergence to equilibrium:

Proposition 6.1 *Let c be a solution to the DCF equations given by Proposition 2.2 under Hypotheses 1–6. Suppose that the mass ρ of the solution c is strictly less than the critical mass ρ_s , and that $M_{2-\lambda}(0) := \sum_{i \geq 1} i^{2-\lambda} c_i(0) < +\infty$. Then for some constant C depending only on the coefficients $a_{i,j}, b_{i,j}$, on $M_{2-\lambda}(0)$ and on the mass ρ ,*

$$F_z(t) \leq \min \left\{ F_z(0), \frac{C}{1 + \log(1 + t)} \right\} \text{ for all } t > 0, \tag{102}$$

where z is such that

$$\sum_{i \geq 1} i Q_i z = \rho. \tag{103}$$

Remark 6.2 By inequality (26), this implies that (for some other C depending on the same quantities)

$$\sum_{i \geq 1} i |c_i - z^i Q_i| \leq \frac{C}{\sqrt{1 + \log t}} \text{ for all } t > 0. \tag{104}$$

This rate is by no means expected to be optimal; in fact, one would expect the solution to converge to equilibrium at a rate similar to the one obtained by Jabin and Niethammer in [16], this is, a convergence like $e^{-Ct^{1/3}}$ for some constant $C > 0$. However, further estimates on the solution (such as, for example, uniform estimates on exponential moments) which are not readily available here are essential in [16] in order to show such convergence.

In order to prove Proposition 6.1 we will use Proposition 5.1 and the following lemma from [16, Lemma 3.6], which is also applicable in our case:

Lemma 6.3 Assume hypotheses 1–4. Let $c = \{c_i\}_{i \geq 1}$ be a nonnegative sequence with mass $\rho < \rho_s$ and such that $c_1 \geq z_s - \frac{1}{4}(z_s - z)$, where z is such that

$$\sum_{i \geq 1} i Q_i z^i = \rho, \tag{105}$$

as usual. There exists a constant $C > 0$ which depends only on the $a_{i,j}, b_{i,j}$ and ρ such that

$$D(c) \geq C. \tag{106}$$

Proof of Proposition 6.1 As before, C, C_1, C_2, \dots are used to denote constants which depend on the quantities stated in the proposition, which will be called “allowed constants”.

First, note that F_z is always finite under these conditions. In fact, one can see that $F_z(t)$ is bounded for all $t \geq 0$ by a constant C_1 which depends only on the coefficients and on ρ (equivalently, the free energy $V(t)$ is bounded by such a constant; as mentioned at the beginning of Sect. 2.3, this result can be found in [6, Lemma 4.2 and p. 680], and can also be deduced from Lemmas 7.1, 7.2 and the expression of F_z in (23)). So, with the H-theorem 2.4, one has

$$F_z(t) \leq F_z(0) \leq C_1 \quad \text{for all } t \geq 0. \tag{107}$$

In fact, by the H-theorem we know that

$$\frac{d}{dt} F_z(t) = -D_{CF}(t) \leq -D(t), \tag{108}$$

where D_{CF} and D are the dissipation rates defined in Sect. 2.3. In order to use the inequality in Proposition 5.1 we note that after [16, Lemma 3.8], whenever $0 < c_1 < z_s$ we have

$$F_z(c) \leq F(c) < +\infty, \tag{109}$$

where F is just F_{c_1} (the same used in Proposition 5.1, and defined before it). So, we can use Proposition 5.1 to get a closed equation *only* when $0 < c_1(t) < z_s$; in fact, if we want a bound which is independent of c_1 , we need to use the inequality only for times t for which $\epsilon < c_1(t) < z_s - \epsilon$ (see Remark 5.2). Hence, we break the argument in three parts: when $c_1(t)$ is close enough to z , we use Proposition 5.1; when $c_1(t)$ is above this region, we use the bound in Lemma 6.3, which controls the dissipation rate D when c_1 is “supercritical”; and for $c_1(t)$ below this region, we use the inequality in Proposition 3.1, which is weaker than that in Proposition 5.1 but holds uniformly for small c_1 . Let us do this:

1. At any time t at which $c_1(t) \geq z_s - \frac{1}{4}(z_s - z)$, Lemma 6.3 shows that for some allowed $C_2 > 0$,

$$D(t) \geq C_2. \tag{110}$$

In order to use it below, note that by (107) this can be bounded by

$$D(t) \geq \frac{C_2 F_z^2(t)}{C_1^2 (1+t)} \quad \text{for all } t \geq 0. \tag{111}$$

2. At any time such that $\frac{z}{2} \leq c_1(t) < z_s - \frac{1}{4}(z_s - z)$, Proposition 5.1, and (109) show that

$$F_z(t) \leq F(t) \leq C_3 \max\{\sqrt{D(t)}\sqrt{M_{2-\lambda}(t)}, D(t)\}, \tag{112}$$

where C_3 is an allowed constant (which bounds the constant called C in Proposition 5.1 for the c_1 under consideration). Then,

$$D(t) \geq \min \left\{ \frac{1}{C_3^2} \frac{F_z^2(t)}{M_{2-\lambda}(t)}, \frac{1}{C_3} F_z(t) \right\}, \tag{113}$$

and with Lemma 4.5,

$$D(t) \geq \min \left\{ \frac{1}{C_4 C_3^2} \frac{F_z^2(t)}{1+t}, \frac{1}{C_3} F_z(t) \right\}, \tag{114}$$

where C_4 is the constant which appears in Lemma 4.5 (called C there). Now, using (107),

$$\frac{F_z}{C_3} \geq \frac{F_z^2}{C_3 C_1} \geq \frac{F_z^2}{C_3 C_1 (1+t)}, \tag{115}$$

so from (114) we have, for some allowed $C_5 > 0$,

$$D(t) \geq C_5 \frac{F_z^2(t)}{1+t}. \tag{116}$$

3. At any time such that $c_1(t) < \frac{z}{2}$, the inequality in Proposition 3.1 and again Lemma 4.5 show that for some allowed constants C_6, C_7 ,

$$C_6 \leq \sum_{i \geq 1} i c_i(t) - \sum_{i \geq 1} i Q_i c_1(t)^i \leq C_7 \sqrt{D(t)} \sqrt{M_{2-\lambda}(t)}, \tag{117}$$

so, with $C_8 := (C_6/C_7)^2$,

$$D(t) \geq C_8 \frac{1}{M_{2-\lambda}(t)} \geq \frac{C_8}{C_4 C_1^2} \frac{F_z^2(t)}{1+t}, \tag{118}$$

where C_1 appears in (107) and C_4 is again the constant in Lemma 4.5.

Hence, gathering (111), (116) and (118), we know that there is an allowed constant C_9 such that

$$\frac{d}{dt} F_z(t) \leq -D(t) \leq -C_9 \frac{F_z^2(t)}{1+t} \quad \text{for all } t \geq 0. \tag{119}$$

Solving this differential inequality proves the proposition. □

Acknowledgements I wish to thank Stéphane Mischler and Clément Mouhot for some discussions concerning these results, which were extremely helpful. The author was supported by the ANR research group *SPINADA* and the University Paris-Dauphine.

Appendix Proof of the H-theorem

In this section we give the proof of the H-theorem 2.4. The usual strategy to prove this result is to calculate the time derivative of an approximation to V for which we know how to do it, and then show that the limit behavior of these approximations imply the H-theorem for V . We will follow this idea in a way similar to the proof of [11, Theorem 5.2]. On the way, we will make use of some simple bounds stated in the following two lemmas:

Lemma 7.1 Take $m > k \in \mathbb{R}$ with $m \geq 1$. For any nonnegative sequence $\{c_i\}$ there is a constant C which depends only on k, m and $M := \sum_{i=1}^\infty i^m c_i$ such that

$$\sum_{i=1}^\infty i^k c_i |\log c_i| \leq C. \tag{120}$$

Proof For any $0 < \epsilon < 1$ there is a constant $C_\epsilon \geq 0$ such that $|x \log x| \leq C_\epsilon(x^{1-\epsilon} + x^{1+\epsilon})$, so we have

$$\sum_{i=1}^\infty i^k c_i |\log c_i| \leq C_\epsilon \left(\sum_{i=1}^\infty i^k c_i^{1-\epsilon} + \sum_{i=1}^\infty i^k c_i^{1+\epsilon} \right). \tag{121}$$

As $c_i \leq M$ for all $i \geq 1$, the second sum is less than $M \sum_{i=1}^\infty i^k c_i \leq M^2$. For the first sum, using Hölder’s inequality with exponents $p = 1/(1 - \epsilon)$, $q = 1/\epsilon$,

$$\sum_{i=1}^\infty i^k c_i^{1-\epsilon} \leq \left(\sum_{i=1}^\infty i^m c_i \right)^{1-\epsilon} \left(\sum_{i=1}^\infty i^{\frac{k-m(1-\epsilon)}{\epsilon}} \right)^\epsilon. \tag{122}$$

As $k < m$, we can choose $\epsilon > 0$ small enough such that the exponent of i inside the second sum is less than -1 ; with such an ϵ , the sum is finite and the result is proved. \square

Lemma 7.2 Take a strictly positive sequence $\{Q_i\}$ such that $C_1 \geq Q_i^{1/i} \geq C_2$ for some $C_1 \geq C_2 > 0$ and all $i \geq 1$. Then for any nonnegative sequence $\{c_i\}$

$$\sum_{i=1}^\infty i^k c_i |\log Q_i| \leq C \sum_{i=1}^\infty i^{k+1} c_i, \tag{123}$$

with $C := \max\{|\log C_1|, |\log C_2|\}$.

Proof One just writes $|\log Q_i| = i |\log Q_i^{1/i}|$ and use the bounds assumed in the lemma. \square

Let us prove Theorem 2.4. First, note that the hypotheses that $a_{i,1} > 0$ for $i \geq 1$ implies that $c_i(t)$ is strictly positive for all $t > 0$ and all $i \geq 1$ [10, 11], so that D_{CF} makes sense for all positive times. Note also that moments which are finite at $t = 0$ remain finite for all times [5]; in particular, the moments of order $1 + \lambda$ and $1 + \gamma$ are always finite under our assumptions. Call

$$V_N := \sum_{i=1}^N c_i \left(\log \frac{c_i}{Q_i} - 1 \right) \quad \text{for } t \geq 0 \text{ and } N \geq 1 \tag{124}$$

and

$$D_{i,j} := W_{i,j} \left(\log \frac{c_i c_j}{Q_i Q_j} - \log \frac{c_{i+j}}{Q_{i+j}} \right) \geq 0 \quad \text{for } t > 0 \text{ and } i, j \geq 1, \tag{125}$$

where the $W_{i,j}$ were defined in (2) as $W_{i,j} := a_{i,j} c_i c_j - b_{i,j} c_{i+j}$, and the time dependence is implied. Take $T > 0$. Then, for any $N \geq 1$, calculating the time derivative of V_N from the

DCF equations (1) gives

$$\begin{aligned}
 V_N(T) - V_N(0) &= -\frac{1}{2} \int_0^T \sum_{i+j \leq N} D_{i,j}(t) dt \\
 &\quad - \int_0^T \sum_{i=1}^N \sum_{j=N-i+1}^{\infty} W_{i,j}(t) \log \frac{c_i(t)}{Q_i} dt, \tag{126}
 \end{aligned}$$

which can be obtained by a direct calculation after differentiating V_N , as the sum defining it has only a finite number of terms. As V is finite, $\lim_{N \rightarrow \infty} V_N(T) = V(T)$, so the result is proved if we can show that the right hand side of the above equality converges to $\int_0^T D_{CF}(t) dt$ as $N \rightarrow \infty$. To do that, let us first show that D_{CF} is locally integrable.

Let us find an upper bound for the rightmost term in (126). It holds that

$$-W_{i,j} \log \frac{c_i}{Q_i} = (b_{i,j}c_{i+j} - a_{i,j}c_i c_j) \log \frac{c_i}{Q_i} \tag{127}$$

$$\leq C i b_{i,j} c_{i+j} + a_{i,j} c_i c_j \left| \log \frac{c_i}{Q_i} \right| =: S_{i,j} \tag{128}$$

for some constant $C \geq 0$ which depends only on ρ and the Q_i , thanks to Hypothesis 3 and the fact that $c_i \leq \rho$ for all i . We have that

$$\sum_{i,j \geq 1} a_{i,j} c_i(t) c_j(t) \left| \log \frac{c_i(t)}{Q_i} \right| \leq C \quad \text{for all } t \in [0, T] \tag{129}$$

for some $C \geq 0$, thanks to Lemmas 7.1, 7.2, our assumption on moments, and Hypotheses 1 and 3. Also, using the bound of $b_{i,j}$ in Hypothesis 1,

$$\sum_{i,j \geq 1} i b_{i,j} c_{i+j} = \sum_{i=1}^{\infty} \sum_{j=i+1}^{\infty} c_j i b_{i,j-i} = \sum_{j=2}^{\infty} c_j \sum_{i=1}^{j-1} i b_{i,j-i} \leq K \sum_{j=2}^{\infty} c_j j^{\gamma+1}, \tag{130}$$

which is again uniformly bounded on $[0, T]$. Together with (129), this proves that $\sum_{i,j} S_{i,j}$ is uniformly bounded on $[0, T]$, and in particular that

$$-\sum_{i=1}^N \sum_{j=N-i+1}^{\infty} W_{i,j}(t) \log \frac{c_i(t)}{Q_i} \leq C \quad \text{for all } t \in [0, T] \tag{131}$$

for some $C \geq 0$. Then, from (126) we deduce that

$$\int_0^T \sum_{i+j \leq N} D_{i,j}(t) dt \leq C \quad \text{for all } N \geq 1 \tag{132}$$

for some other constant C , as $|V_N(t)|$ is uniformly bounded for all times $t \geq 0$. Hence, as $\frac{1}{2} \sum_{i+j \leq N} D_{i,j}(t)$ converges increasingly to $D_{CF}(t)$ as $N \rightarrow \infty$, the monotone convergence theorem shows that D_{CF} is integrable on $[0, T]$ and that

$$\frac{1}{2} \int_0^T \sum_{i+j \leq N} D_{i,j}(t) dt \rightarrow \int_0^T D_{CF}(t) dt \quad \text{when } N \rightarrow \infty. \tag{133}$$

The previous calculations show that

$$-W_{i,j}(s) \log \frac{c_i(s)}{Q_i} \leq S_{i,j}(s) \tag{134}$$

for some $S_{i,j} \geq 0$ such that $\int_0^T \sum_{i,j \geq 1} S_{i,j}(s) ds < +\infty$. Similarly,

$$W_{i,j} \log \frac{c_i}{Q_i} = D_{i,j} - W_{i,j} \log \frac{c_j}{Q_j} + W_{i,j} \log \frac{c_{i+j}}{Q_{i+j}} \tag{135}$$

$$\leq D_{i,j} + S_{j,i} + a_{i,j}c_i c_j \log \frac{c_{i+j}}{Q_{i+j}} - b_{i,j}c_{i+j} \log \frac{c_{i+j}}{Q_{i+j}} \tag{136}$$

$$\leq D_{i,j} + S_{j,i} + C a_{i,j}c_i c_j (i + j) + b_{i,j}c_{i+j} \left| \log \frac{c_{i+j}}{Q_{i+j}} \right| := s_{i,j}, \tag{137}$$

for the same constant C as in (128). Using our previous knowledge that both $\sum_{i,j} D_{i,j}$ and $\sum_{i,j} S_{i,j}$ are integrable on $[0, T]$ and a calculation very similar to the one carried out before, one can show that $\sum_{i,j} s_{i,j}$ is also integrable on $[0, T]$. Then, the dominated convergence theorem proves that the last term in (126) converges to 0 as $N \rightarrow \infty$, which finishes the proof.

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